# Lévy flights from a continuous-time process

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Lévy flight dynamics can stem from simple random walks in a system whose operational time (number of steps $n$) typically grows superlinearly with physical time $t$. Thus this process is a kind of continuous-time random walk (CTRW), dual to the typical Scher-Montroll model, in which $n$ grows sublinearly with $t$. Models in which Lévy flights emerge due to a temporal subordination allow one easily to discuss the response of a random walker to a weak outer force, which is shown to be nonlinear. On the other hand, the relaxation of an ensemble of such walkers in a harmonic potential follows a simple exponential pattern, and leads to a normal Boltzmann distribution. Mixed models, describing normal CTRW's in superlinear operational time and Lévy flights under the operational time of subdiffusive CTRW's lead to a paradoxical diffusive behavior, similar to the one found in transport on polymer chains. The relaxation to the Boltzmann distribution in such models is slow, and asymptotically follows a power law.

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## I. INTRODUCTION

Random walk processes leading to subdiffusive or superdiffusive behavior are adequate for describing various physical situations. Thus the continuous-time random walk (CTRW) model of Scher and Montroll [1] was a milestone in the understanding of photoconductivity in strongly disordered and glassy semiconductors, while Lévy-flight models [2] are adequate for a description of transport in heterogeneous catalysis [3], self-diffusion in micelle systems [4], reactions and transport in polymer systems under conformational motion [5], transport processes in heterogeneous rocks [6], and the behavior of dynamical systems [7]. Closely related models appeared in a description of economic time series [8]. Lévy-related statistics were observed in hydrodynamic transport [9], and in the motion of gold nanoclusters on graphite [10]. Mixed models were proposed , in which the slow temporal evolution (described by a Scher-Montroll CTRW) is combined with the possibility of Lévy jumps, so that in general both subdiffusion or superdiffusive behavior can arise [11].

The continuous-time random walks first introduced by Montroll and Weiss [12] correspond to a stochastic model in which steps of a simple random walk take place at times $t_i$, following some random process with non-negative increments: $\tau_i = t_i - t_{i-1} \geqslant 0$. In a mathematical language, one states that a CTRW is a process subordinated to a random walk under the operational time defined by the process $\{t_i\}$. It is typically thought that a CTRW scheme alone cannot describe any superdiffusive process, so that the introduction of very long jumps is an inevitable part of building a model leading to a superdiffusive behavior.

Let us first discuss a typical CTRW approach. We consider a one-dimensional situation under which a particle from time to time makes a jump to a neighboring lattice site separated from the initial one by a distance $a$. The time $\tau$ between the two jumps is distributed according to some waiting-time distribution, represented by the probability density function (PDF) $p(\tau)$. If the mean waiting time $\overline{\tau}$ exists, the particle's behavior is diffusive, with a diffusion coefficient $D = a^2/2\overline{\tau}$. If the corresponding moment diverges, the particle's behavior becomes subdiffusive, with $\langle r^2(t) \rangle \propto t^\alpha$, where $\alpha < 1$ depends on the PDF $p(\tau)$. Subdiffusive behavior is indicated by a vanishing of the diffusion coefficient $D$. Within this scheme of seems impossible to obtain any type of superdiffusive behavior unless one allows for infinitely long jumps with $\langle a^2 \rangle \to \infty$. Superdiffusive behavior is indicated by divergence of the diffusion coefficient $D$. If $\langle a^2 \rangle$ remains finite, this can be the case only if $\overline{\tau}$ a vanishes. Since $\tau > 0$ and $\overline{\tau} = \int_0^\infty \tau p(\tau) d\tau$, a vanishing of the mean waiting time means that $p(\tau) = \delta(\tau)$—marginal, degenerate situation.

On the other hand the consideration presented above shows only that the waiting-time distribution is not an adequate tool for a description of superdiffusive CTRW's. In what follows we show that superdiffusive CTRW's with bounded step lengths are just as likely to occur as subdiffusive ones. Our considerations will be rather formal, and do not follow from any particular physical model. On the other hand, the fact that Lévy flights can stem from a process subordinated to simple random walks has many important implications. Thus, as we proceed to show, the fast dynamics of a free process can coexist in such models with simple exponential relaxation to a normal Boltzmann equilibrium distribution, if the behavior of an ensemble of random walkers under a restoring force is considered. This shows that the relation between Lévy dynamics and the nonextensive thermodynamics described by nonclassical entropy functions is much looser than typically assumed.

The combination of superdiffusive Lévy flights with a typical CTRW operational time leads to a paradoxical diffusion behavior, having some parallels to transport in polymer chains. Moreover, the existence of a subordination model leading to Lévy flights can be useful in understanding the statistical implications of the processes described by fractional generalizations of diffusion and Fokker-Planck equations [13–15].

 

This paper is organized as follows: In Sec. II we discuss general properties of subordinated random processes. In Secs. III and IV processes subordinated to symmetric and asymmetric random walks are considered, these leading to symmetric and asymmetric Lévy flights. The dualism between the Lévy flights and the Scher-Montroll CTRW is discussed in Sec. V. Sections VI and VII discuss the models leading to paradoxical diffusion behavior. The relaxation to equilibrium is considered in Sec. VIII.

## II. SUBORDINATION OF RANDOM PROCESSES

As already mentioned, a Scher-Montroll CTRW process is a simple random walk whose steps take place at times $t_i$ governed by a random process with non-negative independent increments, so that

$$P(x,t)=\sum_n P_{RW}(x,n)p_n(t), \tag{1}$$

where $P_{RW}(x,n)$ is a probability distribution to find a random walker at point $x$ after $n$ steps (i.e., the binomial distribution), and $p_n(t)$ is the probability of making exactly $n$ steps up to time $t$. For both $t$ and $n$ large, when the binomial distribution can be approximated by a Gaussian one, and when the corresponding sum can be changed to an integral, Eq. (1) reads

$$P(x,t)\simeq\int_0^\infty \frac{1}{\sqrt{2\pi n}}\exp\left(-\frac{x^2}{2n}\right)p_t(n,t)dn. \tag{2}$$

In a classical Scher-Montroll CTRW, $p_t(n,t)$ corresponds to a random process in which $n$ typically grows sublinearly in $t$. Thus the overall process is subdiffusive.

Note that the description of the CTRW process given by Eq. (2) is an example of *subordination*; see Sec. X 7 of Ref. [16]: If $\{X(T)\}$ is a Markov process with continuous transition probabilities and $\{T(t)\}$ a process with non-negative independent increments, then $\{X(T(t))\}$ is said to subordinate to $\{X(t)\}$ using the operational time $T$. In this case,

$$P(x,t)=\int_0^\infty P_x(x,T)p_T(T,t)dT. \tag{3}$$

In what follows we call the integral transform [Eq. (3)] a subordination transformation, changing from time scale $t$ to a time scale $T$. For example, in the Scher-Montroll case the operational time of a system is given by the number of steps of the RW, and is a random function of the physical time $t$ whose typical value grows sublinearly in $t$.

The operational time can also grow superlinearly with $t$. Such a process can not be described by a waiting-time distribution, and needs a complimentary description. Let us consider a random process, where the *density* of events fluctuates strongly. Let us subdivide the time axis into intervals of duration $\Delta t$, and let us consider the number $n$ of jumping events within each interval. The value $\rho=n/\Delta t$ defines the density of jump events. Now, if the mean density of events exists, its inverse gives us exactly the mean waiting time of a jump, and a process described by a finite density of events is a normal diffusive one. The divergence of a mean waiting time (as in a Scher-Montroll CTRW) corresponds to a vanishing density. On the other hand, if one considers a strongly fluctuating density $\rho(t)$ whose first moment diverges, the mean waiting time vanishes, and a process that subordinates a random walk process under such an operational time can be superdiffusive. At longer times, the distribution of the number of events tends to one of the Lévy-stable laws: the typical number of events can grow superlinearly in time. A simple example of such a process was already known to Feller; see Chap. X 7 of Ref. [16]. He considered a process subordinated to simple random walks under the operational time governed by a fully asymmetric Lévy stable law of index 1/2. The corresponding PDF at time $t$ is given by

$$P(x,t)=\int_0^\infty \frac{1}{\sqrt{2\pi n}}\exp\left(-\frac{x^2}{2n}\right)\frac{t}{\sqrt{\pi}n^{3/2}}\exp\left(-\frac{t^2}{2n}\right)dn$$
$$=\frac{t}{\pi(t^2+x^2)}, \tag{4}$$

i.e., it is a Cauchy Lévy-flight.

Let us now discuss a simple analogy describing the relation between the Scher-Montroll CTRW and Lévy flights. This analogy makes clear many of the findings we are going to discuss below. Imagine a physical clock producing ticks following with frequency 1, which govern the behavior of a random walker. Imagine a switch situated at 0, so that, returning to the origin, the walker can trigger some physical process [the analogy with the Glarum model of relaxation (see Ref. [17]), is evident]. The times between the subsequent returns are distributed according to a fully asymmetric Lévy stable law of index 1/2 used in a previous example. Now imagine another random walker performing its motion (a step per physical unit time) independently from the first one. Imagine a movie camera, taking frame-per-frame pictures of the positions of this second random walker at the moments when the first walker is at the origin and thus triggers the switch. Watching the movie taken by the camera, we immediately recognize that the second walker performs the Cauchy Lévy flights. Imagine that a clock is posed in a frame and also filmed. In this case its image will show exactly the operational time of the system; the spectator's watch measures the physical time. Imagine an opposite situation: the first walker triggers the motion of the second one, and the camera is triggered by the physical clock, as a normal movie camera is. The process we recognize at the film is then the Scher-Montroll CTRW. We can also take a Scher-Montroll movie using another trick (which cannot be performed in a real time, but needs a record of return times). Let us take a record of subsequent return times of a first random walker (numbers $n_1$, $n_2$, . . . ), and trigger our camera in such a way that it makes $n_1$ frames during the first second, $n_2$ frames during the second one, etc. If we film a normal random walker with a camera prepared in such a way, the movie will show us a Montroll-Weiss CTRW. An image of the physical clock will again show the operational time of the system, and again, looking at his watch, the spectator can measure the physical time between two events.

Let us use our camera, triggered by returns of a random walker to film other processes taking place in the outer world. The film, which is watched afterwards at a constant speed, shows us a *possible* world: The causality relations and thermodynamical time arrow are those of our usual world. On the other hand, a movie of a world undergoing a continuous evolution, in which "*natura non facit saltus*" holds, will show us a revolutionary world of "great leaps" and abrupt changes (but following the same logics of development). The second camera (fed by a prescribed $n$ sequence) will show us a world of almost full stagnation seldomly interrupted by a bounded, local movement, a world developing in a slow time of old Asiatic despoty. We shall keep this analogy in mind when discussing the physical implications of subordination.

Let us consider a system which evolves according to a Markovian dynamics, and whose state tends to a normal Boltzmann equilibrium under relaxation. In a system under action of outer forces, the transition probabilities between the states of the system (sites $i$ between which the random walk takes place), which are characterized by their energies $E_i$, are not independent. They are connected through the corresponding Boltzmann factors, so that in equilibrium during any period of time $\Delta t$ the mean numbers of forward and backward jumps between any two sites $i$ and $j$ fulfill the condition

$$n_{ij}(\Delta t)/n_{ji}(\Delta t)=\exp[(E_i-E_j)/kT], \tag{5}$$

where $k$ is the Boltzmann constant and $T$ is the system's temperature. Condition (5) guarantees a detailed balance in equilibrium, independent of what the real dynamics of a system is. For simple RW's, where only transitions between the neighboring states are allowed, corresponding transition rates with respect to the operational time of the system can be introduced. For a random walker moving under the influence of a weak constant force $F$ the probabilities of forward and backward jumps per unit times $w_+$ and $w_-$ are connected by $w_+/w_-=\exp(Fa/kT)$. The Markovian nature of RW's then leads to the fact that the values of $w_+$ and $w_-$ do not depend on whether the system is in equilibrium or not. For $F$ small, one can take, say, $w_+=w_0(1+Fa/kT)$ and $w_-=w_0(1-Fa/kT)$ with $w_0=1/2\tau$.

Note that subordination, describing a transition from a physical time to an operational time of the system, does not change its equilibrium properties. Such subordination can be considered as a random modulation of the transition rate $w_0$ by some independent process (say closing and opening the channels), and is fully irrelevant for the thermodynamics (i.e. thermostatics) of the system. On the other hand, it strongly influences its kinetics, so that a question can be posed about what kinds of kinetics are compatible with relaxation to a normal Boltzmann distribution under an arbitrary subordination transformation of time. We address this question in Sec. VIII, after the free diffusion properties of superdiffusive CTRW's are discussed.

## III. SYMMETRIC LÉVY FLIGHTS FROM CTRW'S

Let us first concentrate on the symmetric random walk case. Let us consider a random process in which the number of events per given time is unbounded and follows, for example, a power-law distribution $p_n(t)\propto tn^{-1-\alpha}$, with $0<\alpha\leqslant 1$ (this corresponds to the typical number of events scaling as $n\propto t^{1/\alpha}$). Let us find the asymptotic behavior of $P(x,t)$ for $t$ large. Since the jumps during different intervals are uncorrelated, the PDF of $n$ for longer times converges to a fully asymmetric Lévy stable law

$$p(n,t)\simeq t^{-1/\alpha}L(n/t^{1/\alpha};\alpha,\gamma), \tag{6}$$

with the asymmetry parameter $\gamma=-\alpha$ [here the values of $\gamma=\pm\alpha$ correspond to the strongly asymmetric PDF that vanish identically for large positive (negative) $x$ values, while $\gamma=0$ corresponds to symmetric distributions; the notation is from Ref. [16]]. Note that the Fourier transforms of Lévy-stable laws are known: up to the translation and scaling $P(k,t)$ is equal to

$$f(\kappa)=\exp[-|\kappa|^a e^{i\pi\gamma/2}] \tag{7}$$

(for $0<\alpha<2$, $\alpha\neq 1$). The PDF is a real function, thus $f(\kappa)=f^*(-\kappa)$. The corresponding function is analytical everywhere except for $\kappa=0$, so that the PDF is given by

$$L(x;\alpha,\gamma)=\frac{1}{\pi}\mathrm{Re}\int_0^\infty e^{-ix\zeta-\zeta^\alpha e^{i\pi\gamma/2}}d\zeta. \tag{8}$$

From Eq. (8) the series expansions for $L(y;\alpha,\gamma)$ follow, see Sec. XVII 6 of Ref. [16]. In the case $\alpha<1$ one can move the path of integration to the negative imaginary axis (since the integrand tends to zero when $\mathrm{Im}\,\zeta\to-\infty$ due to the dominance of the linear term), which then allows for elementary integration after a Taylor expansion of $\exp(A\zeta^\alpha)$. For $1<\alpha<2$ this dominance is no longer the case, but the integrand still vanishes for $\mathrm{Im}\,\zeta\to-\infty$ in the case of symmetric distributions, while $(-i|\zeta|)^\alpha=|\zeta|^a[\cos(\pi/2)\alpha-i\sin(\pi/2)\alpha]\to-\infty$ for $1<\alpha<2$. Thus the series which represents Lévy distributions for $0<\alpha<1$, and also a symmetric Lévy distribution for $1<\alpha<2$, $\gamma=0$, reads

$$L(y;\alpha,\gamma)=\frac{1}{\pi y}\sum_{k=1}^{\infty}(-1)^k\frac{\Gamma(k\alpha+1)}{k!}\times\sin\left(\frac{k\pi}{2}(\gamma-\alpha)\right)y^{-\alpha k} \tag{9}$$

(in the last case the series does not converge absolutely). In general, the Lévy-stable laws for $1<\alpha<2$ are given by another expansion,

$$L(y;\alpha,\gamma)=\frac{1}{\pi y}\sum_{k=1}^{\infty}(-1)^k\frac{\Gamma(1+k/\alpha)}{k!}\sin\left(\frac{k\pi}{2}(\gamma-\alpha)\right)y^{-k}, \tag{10}$$

which also holds for asymmetric laws.

One can easily obtain the form of the $x$ distributions by immediate integration: using Eq. (2) and a scaling form of a Lévy distribution,

$$p(x,t) \simeq \int_0^\infty \frac{1}{\sqrt{2\pi n}} \exp\left(-\frac{x^2}{2n}\right) L\left(\frac{n}{t^{1/\alpha}};\alpha,-\alpha\right)\frac{dn}{t^{1/\alpha}}. \tag{11}$$

Using Eq. (9) and performing a term-by-term integration, we arrive at a series of integrals of the form

$$I_\mu(\zeta) = \int_0^\infty \frac{1}{\sqrt{2\pi\xi}} e^{-(\zeta^2/2\xi)}\xi^{-\mu}d\xi = \frac{1}{\sqrt{2\pi}}\left(\frac{\zeta^2}{2}\right)^{1/2-\mu}\Gamma(\mu-1/2). \tag{12}$$

For an integral of the $k$th term in Eq. (9), we have $\mu = 1 + \alpha k$ . Let us first concentrate on the case $0<\alpha<1$. Using well-known relations for the $\Gamma$ function, $\Gamma(z+1)=z\Gamma(z)$ [Eq. (6.1.15) of Ref. [18]] and $\Gamma(2z) = (2\pi)^{-1/2}2^{2z-1/2}\Gamma(z)\Gamma(z+1/2)$ [Eq. (6.1.18) of Ref. [18]], we obtain

$$p(\zeta) = \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^k\frac{\Gamma(2k\alpha+1)}{k!}\sin(-k\pi\alpha)\left(\frac{\sqrt{2}}{\zeta}\right)^{-2\alpha-1}, \tag{13}$$

which represents a series expansion for a symmetric Lévy-stable law of index $2\alpha$ [Eq. (9)], for the scaled variable $\zeta/\sqrt{2}$. This corresponds to a form $p(x,t) = t^{-1/2\alpha}L(x/\sqrt{2}t^{2\alpha};2\alpha,0)$ of the $x$-distribution.

We note that taking the Fourier transform of both parts of the equation for symmetric RW's,

$$L(ax,2\alpha,0) = \int_0^\infty \frac{1}{\sqrt{2\pi n}}\exp\left(-\frac{x^2}{2n}\right)L(n;\alpha,-\alpha)dn, \tag{14}$$

where $a$ is an unimportant scaling factor, we obtain:

$$\exp(-A|k|^{2\alpha}) = \int_0^\infty \exp(-k^2 n)L(n;\alpha,-\alpha)dn, \tag{15}$$

which holds for any real $k$ (i.e., for any positive $k^2$), where $A$ is a number factor. This gives us a general expression for a Laplace transform of an asymmetric Lévy distribution with $\alpha<1$: a Laplace transform of $L(n,\alpha,-\alpha)$ is $\exp(-Au^\alpha t)$. From this fact an important result follows:

$$L(ax;\alpha\beta,0) = \int_0^\infty n^{-1/\beta}L(x/n^{1/\beta};\beta,0)L(n;\alpha,-\alpha)dn. \tag{16}$$

A Lévy distribution with index $\alpha\beta$ is subordinated to a Lévy distribution with index $\beta<\alpha$ under the operational time given by an asymmetric Lévy law of index $\alpha<1$. To see this, consider the characteristic functions of both sides of Eq. (16), and use Eq. (15).

$$\exp(-A|k|^{\alpha\beta}) = \int_0^\infty e^{-|\kappa|^\beta n}L(n;\alpha,-\alpha)dn, \tag{17}$$

see Sec. X 7 of Ref. [16]. Equation (15) corresponds to a special case of $\beta=2$ of Eq. (17). The distributions $L(n;\alpha,-\alpha)$ thus coincides with inverse Laplace transforms of stretched exponentials. For example, for $L(n;1/2,-1/2)$ one readily obtains:

$$p(n,t) = \mathcal{L}^{-1}\{\exp(-tu^{1/2})\} = \frac{t}{2\sqrt{\pi}n^{3/2}}\exp\left(-\frac{t^2}{4n}\right), \tag{18}$$

which differs only by a scale for the time-unit from a distribution used in the example of Eq. (4).

## IV. ASYMMETRIC LÉVY FLIGHTS

Imagine a random walker moving under the influence of a weak constant force $F$. Such a force introduces an asymmetry into the walker's motion, since the probabilities of the forward and backward jumps $w_+$ and $w_-$ are now weighed with the corresponding Boltzmann factors, $w_+/w_- = \exp(Fa/kT)$. For $F$ small one can take $w_+ = 1/2 + Fa/2\tau kT$ and $w_+ = 1/2 + Fa/2\tau kT$. For $t$ large such random walks lead to a Gaussian distribution of the particles' positions,

$$P_{RW}(x,t) = \frac{1}{\sqrt{2\pi n}}\exp\left(-\frac{(x-vn)^2}{2n}\right), \tag{19}$$

whose center moves with a constant velocity $v = \mu F = Fa^2/2\tau kT$. Note that our RW's fulfill Einstein's relation between the mobility $\mu$ and diffusion coefficient $D$: $\mu = D/kT$. The PDF of a random process which subordinates biased RW's under an operational time following the asymmetric Lévy law is given by

$$P(x,t) \simeq \int_0^\infty \frac{1}{\sqrt{2\pi n}}\exp\left(-\frac{(x-vn)^2}{2n}\right)L\left(\frac{n}{t^{1/\alpha}};\alpha,-\alpha\right)\frac{dn}{t^{1/\alpha}}. \tag{20}$$

Using the series expansion [Eq. (9)] and performing a term-by-term integration, leads to a series of the integrals of the type

$$I_\mu(\zeta,\omega) = \int_0^\infty \frac{1}{\sqrt{2\pi\xi}}e^{-(\zeta-\omega\xi)^2/2\xi}\xi^{-\mu}d\xi = \frac{2\exp(\zeta\omega)}{\sqrt{2\pi}}\left(\frac{\zeta^2}{\omega^2}\right)^{1/4-\mu/2}K_{1/2-\mu}(\zeta\omega) \tag{21}$$

for $\omega\neq 0$. For an integral of the $k$th term in Eq. (9), we again have $\mu = 1+\alpha k$. Let us concentrate first on the case $0<\alpha$

$<1$. For $\zeta\omega$ small, $v$ cancels [see expansion (9.6.9) of Ref. [18], $K_\nu(z)\simeq\frac{1}{2}\Gamma(\nu)(\frac{1}{2}z)^{-\nu}$ $(\nu>0)$, note that $K_{-\nu}(z)=K_\nu(z)$], so that the corresponding distribution tends to be a function of $\zeta$ only; it coincides with one for $\omega=0$ [Eq. (12)], so that a symmetric Lévy-stable law of index $2\alpha$ [Eq. (9)] emerges. On the other hand, for $v\neq 0$ and $x$ large the overall distributions follow from the expansion of $K$ for large values of the argument, which reads $K_v(z)\simeq\sqrt{(\pi/2z)}e^{-z}$ [Eq. (9.7.2) of Ref. [18]]. The corresponding integral then tends to $1/\nu(\zeta/\omega)^{-\mu}$, so that the corresponding PDF reproduces the PDF of the density of events (up to rescaling). This last form is also the asymptotic from corresponding to the behavior of Eq. (20) for large $t$.

Hence the distribution $P(x,t)$ tends to a fully asymmetric one of index $\alpha$ for $x$ and $t$ large. In this case the distribution shows scaling with a scaling parameter $\xi=x/(vt)^\alpha$. We see that in this case the motion under the influence of a constant force is superdiffusive, so that $x\simeq(Ft)^{1/\alpha}$, and its dependence on the outer force is nonlinear. Thus the model shows a behavior that differs considerably from the linear response assumption of Refs. [11,19,20]. This absence of a linear response regime is parallel to the CTRW findings [1] (see Ref. [21] for a review) and mirrors the fact that only for normal diffusion a sweep with constant velocity and a drift under a constant force result in the same pattern of motion, see Ref. [14].

The case $\alpha=1/2$ again results in a closed expression:

$$P(x,t)=\int_0^\infty \frac{1}{\sqrt{2\pi n}}\exp\left(-\frac{(x-vn)^2}{2n}\right)\frac{t}{\sqrt{2\pi n^3}}\exp\left(-\frac{t^2}{2n}\right)dn$$

$$=\frac{1}{\pi}\frac{vt}{\sqrt{x^2+t^2}}e^{vx}K_1[\sqrt{v^2(x^2+t^2)}] \tag{22}$$

[Eq. (2.3.16.1) of Ref. [25]]. For $v,x$, and $t$ small, the corresponding distribution tends to a Cauchy law. On the other hand, for $t$ large we can take approximately

$$P(x,t)\approx\frac{1}{\sqrt{2\pi}}\frac{\sqrt{v}t}{(x^2+t^2)^{3/4}}e^{v(x-\sqrt{x^2+t^2})}. \tag{23}$$

The second moment of this distribution diverges, but the position of the maximum of $P(x,t)$, determining the typical particle position at time $t$, tends to grow as $x_{\max}=\frac{2}{3}t^2$ for $t$ large. Thus the typical behavior of $x(t)$ under a constant force is superlinear.

Note that in the case $1<\alpha<2$ the distribution of the particle's displacement for the case $v=0$ will tend to a Gaussian, but in the case $v>0$ it still tends to a fully asymmetric Lévy one. On the other hand, in this case the distribution of the particle's position possesses the first moment which grows linearly with time, thus the situation under $\alpha>1$ shows a linear response behavior. Since the second moment of the distribution is absent, the fluctuations are strong, and the width of such distribution is of the order of the typical value of $x$ itself.

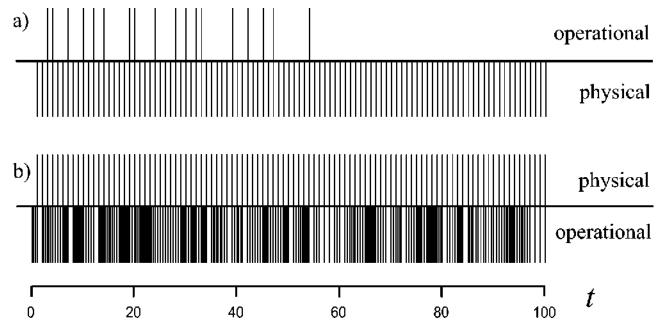


FIG. 1. This figure illustrates the notion of operational time. (a) Operational time leading to a Scher-Montroll CTRW. The ''ticks'' triggering the motion of the random walker (shown as bars above the horizontal line) are taken from the set of ticks of a physical clock according to a waiting-time distribution $\psi(n)\propto(n+1)^{-1-\gamma}$ (shown as bars below the line). The process shows long lacunae; see the text for details. (b) We now declare each interval between the two moves of the random walker for a new time unit. The initial ticks of the physical clock now follow extremely inhomogeneously and show intervals of high condensation. This kind of operational time leads to Lévy flights.

## V. DUALISM BETWEEN SUBDIFFUSIVE AND SUPERDIFFUSIVE CTRW'S

There exists a clear dualism between a normal, subdiffusive CTRW and a superdiffusive one. The corresponding concepts are illustrated in discrete time by Fig. 1, where we return to a situation discussed in Sec. II. Imagine a clock producing ticks following with frequency 1, marking the physical time of a system. Imagine a system which is triggered not by each tick of a physical clock, but follows some waiting-time distribution $\psi(n)$. This means that after our random walker has jumped, the next jump will take place after $n$ ticks of a clock, where the number $n$ is chosen according to a power-law distribution, say $\psi(n)\propto(n+1)^{-1-\gamma}$. The number $n$ fluctuates strongly, so that the sequence of jumps (corresponding to a randomly decimated sequence of ticks) shows lacunae of different duration. Figure 1(a) shows a realization of such a sequence for the case $\gamma=0.75$. The lacuna starting in the middle of Fig. 1(a) at $t=54$ ends at $t=161$. The mean number of jumps during the time $t$ grows sublinearly with $t$, namely, as $t^{3/4}$. Let us denote the corresponding subordination transformation as a time-expanding transformation (TET) of index $\gamma$. According to the procedure described above, the corresponding sequence does not have any intervals where the density of events is larger than 1. The process, subordinated to random walks under such an operational time (a normal CTRW), is subdiffusive.

Let us now consider a sequence of jumps of a walker as ticks marking relevant time epochs of a system (i.e., associate each jump with a tick of a physical clock). From this point of view, the ticks of initial clocks follow extremely inhomogeneously, so that the number of such ticks within a physical time unit varies according to $p(n)\propto(n+1)^{-1-\gamma}$. Figure 1(b) illustrates this situation: Here we took 100 jumps from the realization shown in Fig. 1(a), and rescaled each of the corresponding time intervals to the unit length. The ticks

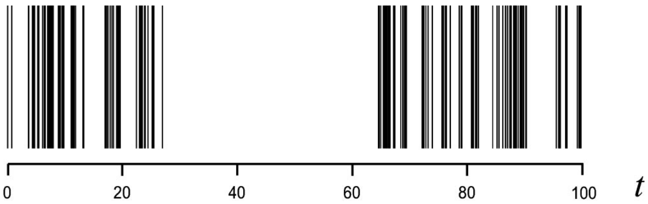


FIG. 2. The operational time stemming from subordination of the two processes depicted in Fig. 1. Note that the bar-code-like set shows both intervals of high condensation and long lacunae.

of initial clock (shown as bars) follow inhomogeneously, and show intervals of high concentration (but no lacunae). The number of such events grows superlinearly in time. The corresponding subordination transformation will be called a ''time-squeezing transformation'' (TST) of index $\gamma$. The process, subordinated to random walks under such an operational time, is superdiffusive, and corresponds to Lévy flights. Note that both the TST and TET are probability distributions $P(n,t)$ of the operational time $n$ for a given physical time $t$, i.e., they are positive, integrable functions of $n$.

Let us now combine the two processes. For example, let us first generate a superlinear sequence using the algorithm described above [with $p(n)\propto(n+1)^{-1-\gamma}$], and then decimate it randomly according to the waiting-time distribution $\psi(n)\propto(n+1)^{-1-\gamma}$. In this case the typical number of events during a time interval $t$ grows linearly with $t$, but the corresponding sequence of events is extremely inhomogeneous, showing both lacunae and accumulation intervals on all scales. This process is shown in a bar-code-like picture in Fig. 2. This will be discussed in more detail in Secs. VI and VII. We can also proceed in other way, and apply the transformations oppositely, namely, first generating a sublinearly growing, lacunary operational time and then filling the lacunae according to a Lévy distribution. As we proceed to show, these two ways of constructing the event-time sets are not equivalent. The process, subordinated to a RW under such an inhomogeneous operational time, is a kind of a continuous-time Lévy flight, and not a normal RW.

The example discussed above shows that transformations leading to a sublinear or superlinear operational time behavior (dual to each other in the sense described above) are not the inverse of one another. Let us discuss the possibility of a subordination transformation transforming a Lévy-stable distribution of index $\beta$ (for example, a Gaussian distribution) into one with a distribution of index $\gamma$, in the sense that

$$L(ax;\gamma,0)=\int_0^\infty n^{-1/\beta}L(x/n^{1/\beta};\beta,0)S(n,t)dn, \qquad (24)$$

where $S(n,t)$ is supposed to be a probability distribution of the number of steps $n$ done up to time $t$. Taking Fourier transform of both parts of Eq. (24), and changing to a variable $u=|k|^\beta$, we obtain

$$\exp(-A|u|^\alpha)=\int_0^\infty e^{-un}S(n,t)dn, \qquad (25)$$

with $\alpha=\gamma/\beta$. From Eq. (25) it follows that $S(n,t)$ are the inverse Laplace transforms of the stretched exponential $\exp(-Au^\alpha)$. Note that according to the Bernstein's theorem, a function $f(x)$ is a Laplace transform of a probability distribution if and only if it is completely monotonic [i.e., it is infinitely differentiable and $(-1)^n f^{(n)}(x)\geqslant 0$ for all derivatives $f^{(n)}$], and $f(0)=1$. The last condition is always fulfilled. Note that according to criterion 2 discussed on p. 441 of Vol. II of Ref. [16], a function $f(x)=e^{-\psi(x)}$ is a completely monotonic function if and only if $\psi$ is a positive function with a completely monotonic derivative. In our case $\psi(x)=Au^\alpha$. For $0<\alpha<1$, one has $g(x)=\psi'(x)=A\alpha u^{\alpha-1}>0$, and the higher derivatives (defined on the interval $0<x<\infty$) are $g'(x)=A\alpha(\alpha-1)u^{\alpha-2}<0$, $g''(x)=A\alpha(\alpha-1)(\alpha-2)u^{\alpha-3}>0$, $g'''(x)=A\alpha(\alpha-1)(\alpha-2)(\alpha-3)u^{\alpha-3}<0$, etc., so that $(-1)^n g^{(n)}(x)\geqslant 0$; thus the function $g$ is completely monotone. Thus $S(n,t)$ is a probability distribution (namely the one we have found above by explicit calculation). On the other hand, for $\alpha>1$ the function $g(x)$ is not completely monotonic, so that $S(n,t)$ is *not* a probability distribution. Thus there is no random process which defines the operational time in such a way that the Lévy flight of index $\alpha_1$ will be transformed into a Lévy flight with index $\alpha_2>\alpha_1$. The absence of an inverse of a TST belonging to a class of subordination transformations has a deep physical interpretation: a TST is a coarse-graining procedure (see Fig. 1): information about the internal steps of the process is lost. One cannot anticipate that a transformation inverse to a coarse-graining procedure belongs to the same class as a direct transformation.

Note also that the fact that the TET and TST are not inverses of one another other is mirrored by the fact that within the formalism based on the fractional Fokker-Planck equations (FFPE's), the first one corresponds to an additional fractional *time derivative* on the left-hand side of the FFPEs, while the second one is represented by a fractional *spatial derivative*; see Refs. [11,13,14,19]. Note also that the noncommutativity mentioned above shows that the order of application of these derivatives is fixed and cannot be arbitrarily changed.

## VI. ''PARADOXICAL'' DIFFUSION

A process subordinated to a Lévy CTRW under the TET (a time transform leading to subdiffusive CTRW) was considered in detail in Ref. [11]. We now know that this process subordinates normal random walks under a combination of TST's and TET's of different indices $\beta$ and $\gamma$. The overall behavior of the process is superdiffusive for $\gamma<\beta$ and subdiffusive for $\gamma>\beta$. This is easy to understand since the scaling considerations show that the operational time grows superlinearly with physical time in the first case and that the behavior is sublinear in the second case. Note that the index $\mu$ of the corresponding Lévy flight is exactly $2\beta$, so that this behavior is exactly the one obtained in Ref. [11]. In the case when $\beta=\gamma$ the operational time grows linearly with the physical one: Ref. [11] suggests that it falls into the diffusion universality class. On the other hand this diffusion is a very special one: We will call a process subordinated to RW's under such an operational time paradoxical diffusion. The random process defining an operational time stemming from a combination of TST's and TET's of the same index $\gamma$ has interesting properties: $n$ typically grows in proportion to $t$; on the other hand, neither a well-defined density, nor a well-defined mean waiting-time, exists.

Let us first discuss the situation mentioned in the beginning of the section: a RW subordinated to a Lévy-distributed operational time, driven by a sublinear operational time. The PDF of the corresponding random walks has power-law tails, exactly those of a Lévy-distribution of index $\gamma$. On the other hand, the overall width of the corresponding curve grows as $\Lambda \simeq \sqrt{t}$. Moreover, the whole distribution scales a as a function of dimensionless displacement $\xi = x/\Lambda$: the overall behavior is somewhat similar to one found on a polymer chain with bridges. The overall form of the function can be found using the well-known expression for $p(n,u)$, the Laplace transform of the probability $p(n,t)$, to make exactly $n$ steps up to time $t$. Such a process corresponds to directed motion under the same operational time as a CTRW. For the ordinary renewal process one has $p(n,u) = (1/u)[1-\psi(u)]\psi^n(u)$, with $\psi(u) \simeq 1 - u^\gamma$ [22]. For $u$ small ($t$ large) this form corresponds to

$$p(n,u) \simeq u^{\gamma-1}\exp(-nu^\gamma). \tag{26}$$

Considering paradoxical diffusion as a process subordinated to Lévy flights of index $2\gamma$ under operational time given by $p(n,t)$, we obtain for $P(k,u)$, the Fourier-Laplace transform of $P(x,t)$:

$$P_\gamma(k,u) = \int_0^\infty e^{-|k|^{2\alpha}n} p(n,u)\,dn \simeq \frac{u^{\gamma-1}}{|k|^{2\gamma}+u^\gamma}. \tag{27}$$

The scaling nature of the distribution is immediately evident; the nature of its power-law tails follows from the asymptotic analysis for $k$ small: The tail of $P_\gamma(\xi)$ stems from those of $L(x,2\gamma,0)$, and has a power-law asymptotics $P_\gamma(\xi) \propto \xi^{-1-2\gamma}$ ($\gamma<1$). Note that such a distribution was obtained in Ref. [11] as a solution of a fractional diffusion equation, describing a random process incorporating Lévy jumps taking place under a sublinear operational time. As an example let us consider the distribution $P_{1/2}(x,t)$, i.e., for $\gamma = 1/2$. This distribution has a simple analytical form, which can be obtained by an inverse Laplace-Fourier transformation of Eq. (27). The inverse Laplace transform of Eq. (27) is given in Eq. (3.21) of Ref. [23], and reads $P_{1/2}(k,t) = \exp(k^2 t)\mathrm{erfc}(|k|t^{1/2})$. The inverse (cosine) Fourier transform of this function is given by Eq. (10.6) of Ref. [24], and reads

$$P_{1/2}(x,t) = -\frac{1}{2\sqrt{t}}\pi^{-3/2}\exp(x^2/4t)\,\mathrm{Ei}(-x^2/4t), \tag{28}$$

where $\mathrm{Ei}(x)$ is the exponential integral; see Eq. (5.1.2) of Ref. [18]. The corresponding function is a scaling function of $\xi = x/t^{1/2}$; its behavior for $\xi$ large follows from asymptotic expansion of $-\mathrm{Ei}(-x) = \mathrm{E}_1(x) = x^{-1}e^x[1 - 1/x + \cdots]$, so that asymptotically $P_{1/2}(\xi)$ shows an $\xi^{-2}$-like tail, similar to one of Cauchy-distribution. For $\xi \to 0$ the distribution $P_{1/2}(k,t)$ shows a weak (logarithmic) singularity [following from Eq. (5.1.11) of Ref. [18]], a sign of strong lacunarity of the corresponding operational time. The asymptotic analysis of Eq. (27) shows that such integrable singularities appear in the center of distribution for $0 < \gamma \leqslant 1/2$: the behavior for $\xi \to 0$ is given by $P_\gamma(\xi) \propto \xi^{2\gamma-1}$, for $\gamma = 1/2$ $P_\gamma(\xi)$ diverges logarithmically, as we already saw in Eq. (28).

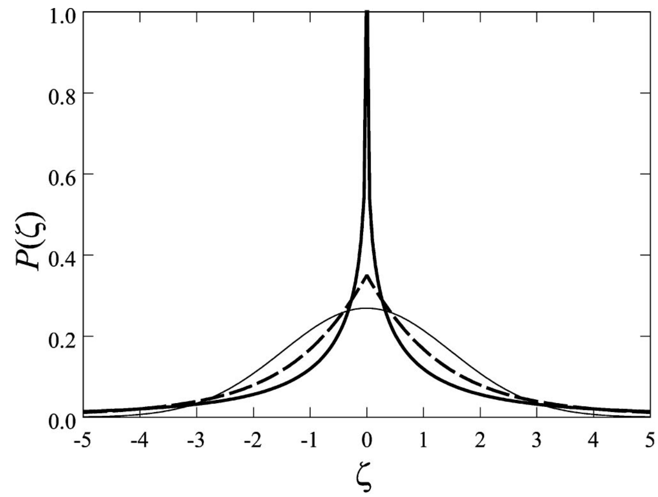


FIG. 3. The PDF of the random walker's positions for paradoxical diffusion. The PDF's are plotted as functions of the dimensionless variable $\zeta = x/Q$, where $Q$ is the position of the upper quartile of the corresponding distribution. The thick full line corresponds to RW under the subordination of a TST and a TET [Eq. (28)]; the dashed line corresponds to the inverse situation [Eq. (34)]. The thin full line represents a Gaussian distribution of the same width.

The distribution $P_{1/2}(\xi)$ is plotted in Fig. 3 together with the Gaussian distribution [i.e., the distribution $P_1(\xi)$ of the same class, the one corresponding to a normal diffusion] and with the distribution stemming from the inverse order of application of TET's and TST's to a simple diffusion, which is discussed in detail in Sec. VII. All distributions are normalized in such a way that their quartiles coincide. Note that the quartiles of $P_{1/2}(\xi)$ are situated at $\pm 0.841$.

## VII. NONCOMMUTATIVITY OF TIME SUBORDINATION

Applying the transformations the other way around, i.e., considering a process subordinated to a Scher-Montroll CTRW under Lévy time, we obtain a process which is different from the one discussed above. Let us start with a simple example.

Let us note that the TET of index 1/2 (corresponding to an inverse Laplace-transform of the function $e^{-n\sqrt{u}}/\sqrt{u}$) is given by

$$Q_{1/2}(n,t) = \frac{1}{\sqrt{\pi t}} e^{-n^2/4t}, \tag{29}$$

i.e., corresponds to part of a Gaussian distribution for $n>0$, so that $n$ typically grows as $t^{1/2}$. The corresponding TST is given by a distribution [Eq. (18)], $R_{1/2}(T,n)$

$=(n/2\sqrt{\pi}T^{3/2})e^{-n^2/4T}$. The subordination of these two processes is described by a function

$$S_{1/2}(T,t)=\int_0^\infty \frac{1}{\sqrt{\pi t}}e^{-n^2/4t}\frac{n}{2\sqrt{\pi}T^{3/2}}e^{-n^2/4T}dn$$
$$=\frac{2}{\pi t}\sqrt{\frac{t}{T}}\left(\frac{T}{t}+1\right)^{-1}, \qquad (30)$$

which is a probability distribution with the tail decaying as $T^{-3/2}$ (as a tail of a stable distribution of index 1/2), and with the square-root singularity at zero. Note that this distribution is just a solution of a fractional Liouville equation describing directed motion under such an operational time, just as Eq. (28) is the solution of a fractional diffusion equation. This is a process subordinated to a Lévy one under sublinear time growth.

We now show that $Q$ and $R$ distributions leading to paradoxical diffusion are not commutative: An operatational time resulting from an $RQ$ transformation has a different distribution from one stemming from a $QR$ one. For example, the distribution $S_{1/2}(T,t)$ given by Eq. (30) is $S_{1/2}(T,t)=Q*R=\int Q(n,t)R(T,n)dn$. Let us calculate a conjugated distribution $S^*_{1/2}(T,t)=R*Q=\int R(n,t)Q(T,n)dn$, describing a process subordinated to a sublinear growth under the operational time growing according to a Lévy distribution. The distribution $S^*_{1/2}(T,t)$ is given by

$$S^*_{1/2}(T,t)=\int_0^\infty \frac{1}{\sqrt{\pi n}}e^{-T^2/4n}\frac{t}{2\sqrt{\pi}n^{3/2}}e^{-t^2/4n}dn=\frac{2t}{\pi}\frac{1}{t^2+T^2}, \qquad (31)$$

i.e., corresponds to the positive part of a Cauchy distribution. Note that even such a robust scaling property of a probability distribution as a nature of its power-law tail is different from one for its conjugated counterpart.

The plausible scaling consideration here is as follows. The distribution $Q(T,n)$ has all moments, so that for $n$ large the value of $T$ is well defined, and is of the order of $n^\alpha$, $\alpha<1$. On the other hand, the distribution of $n$ as a function of $t$ is broad and shows a power-law tail $P(n,t)\propto t^{-1/\alpha}(n/t^{1/\alpha})^{-1-\alpha}\propto tn^{-1-\alpha}$. Now changing the variable from $n$ to $T\propto n^\alpha$, we obtain the asymptotics of the PDF of $T$ in a form $P(T,t)\propto tT^{-2}$, independently of $\alpha$. We thus note that the probability distribution subordinating a sublinear continuous-time directed motion under the Lévy-distributed operational time of the same index has a power-law tail decaying as $T^{-2}$, i.e., is similar to a Cauchy distribution.

The process subordinated to a Gaussian RW, under an operational time defined by $S^*_{1/2}(T,t)$, is also not a normal diffusion, but represents a marginal situation of a distribution whose second moment diverges logarithmically. The corresponding PDF shows power-law tails of a $x^{-3}$ type. This PDF is given by

$$P^*_{1/2}(x,t)=\int_0^\infty \frac{1}{\sqrt{2\pi T}}e^{-x^2/2T}\frac{2t}{\pi}\frac{1}{t^2+T^2}dT. \qquad (32)$$

Changing to a new variable $\zeta=x^2/2T$, and then introducing a scaling variable $\xi=x/\sqrt{t}$, we obtain the PDF $P(x,t)$ as a scaling function of $\xi$:

$$P^*_{1/2}(\xi)=\frac{1}{\pi^{3/2}}|\xi|\int_0^\infty \frac{\zeta^{1/2}e^{-\zeta}}{\zeta^2+\xi^4/4}d\zeta. \qquad (33)$$

For $\xi$ large the corresponding integral decays as $(2/\pi)\xi^{-3}$. Note that Eq. (33) can be expressed in terms of Fresnel sine and cosine integrals $S(x)$ and $C(x)$, so that $P(\xi)$ can be obtained in a closed form:

$$P^*_{1/2}(\xi)=\frac{1}{\sqrt{\pi}}\left\{\sin\left(\frac{\xi^2}{2}\right)\left[1-2S\left(\frac{|\xi|}{\sqrt{\pi}}\right)\right]+\cos\left(\frac{\xi^2}{2}\right)\left[1-2C\left(\frac{|\xi|}{\sqrt{\pi}}\right)\right]\right\}; \qquad (34)$$

see Eq. (2.3.7.10) of Ref. [25]. The corresponding distribution is also plotted in Fig. 3 as a dashed line. Note that the distribution shows a cusp singularity at $\xi=0$. The value of $P(\xi)$ in this point is $1/\sqrt{\pi}=0.564\ldots$. The quartiles of this distribution are situated at $\pm0.621$.

## VIII. RELAXATION PHENOMENA UNDER TEMPORAL SUBORDINATION

The fact that the Lévy dynamics can follow from a temporal subordination is important if one wants to analyze the possible thermodynamical implications of the Lévy-flight transport. Imagine an ensemble of thermodynamical systems (say Brownian particles in a harmonic potential) which was put out of equilibrium and then allowed to relax. As discussed in Sec. II, such relaxation will lead to a stationary state corresponding to a normal equilibrium Boltzmann distribution. Since this distribution is time independent, it would not change under temporal subordination, so that systems with Lévy dynamics may have very ordinary thermodynamical equilibrium states, and thus be described by normal Gibbs-Boltzmann entropy. The non-Boltzmann nature of the equilibrium found in Ref. [19] was connected with the fact that the linear response was considered, as proposed in Ref. [20], an assumption at variance with the findings of Sec. IV. Let us now discuss the relaxation to this equilibrium.

A system slightly outside of equilibrium can be considered as evolving under the influence of the linear restoring force. In the operational time of the system (marked by the number $n$ of jumps) this relaxation will be described by a Fokker-Planck equation. For an overdamped particle in a harmonic potential we obtain, for example,

$$\frac{\partial P}{\partial n}=\frac{\partial}{\partial x}\left(\gamma kxP+D\frac{\partial}{\partial x}P\right). \qquad (35)$$

Note that the values of $\gamma$ and $D$ fulfill Einstein's relation $\gamma=D/kT$. The Green's function of Eq. (35) has a form of a Gaussian distribution, and reads

$$G(x,n|x_0,n_0)=\sqrt{\frac{\gamma}{2\pi D(1-e^{-2\gamma(n-n_0)})}}\times\exp\left(-\frac{\gamma k(x-e^{-\gamma(n-n_0)}x_0)^2}{2D(1-e^{-2\gamma(n-n_0)})}\right); \tag{36}$$

see Sec. 5.4 of Ref. [26]. This equation gives us, e.g., the PDF at a time $n$ in a system, in which the particles were all situated at $x=x_0$ at $n=n_0$. It is easy to see that the first two central moments $M_1=\langle x\rangle$ and $M_2=\langle(x-\langle x\rangle)^2\rangle$ relax exponentially to their equilibrium values, so that

$$\langle x(n)\rangle=x_0\exp(-\tau^{-1}n) \tag{37}$$

and

$$\sigma^2(n)=\frac{D}{k\gamma}[1-\exp(-2\tau^{-1}n)], \tag{38}$$

being a typical pattern of relaxation of a system with only one relaxation time $\tau=(k\gamma)^{-1}$. Since all higher moments of a Gaussian distribution are the combinations of the lower two, they also relax to their equilibrium values in a (multi-)exponential fashion. Let us start from the Fourier-transform of Eq. (36), and note that under subordination

$$P(k,t)=\int\exp[-ikx'e^{-\gamma n}-Dk^2(1-e^{-2\gamma n})/2\gamma]\times t^{-1/\alpha}L(n/t^{1/\alpha},\alpha,-\alpha)dn. \tag{39}$$

Let us moreover expand the exponential term in a Taylor series in $k$: the coefficients of this series give the moments of the corresponding distribution. From Eq. (39) it follows then that the $i$th moment is a combination of integrals of the type

$$\Phi(t)=\int_0^\infty\exp(-\lambda n)t^{-1/\alpha}L(n/t^{1/\alpha},\alpha,-\alpha)dn, \tag{40}$$

with $\lambda=m\gamma$, $0\leqslant m\leqslant i$. Using the fact that a Laplace transform of a fully asymmetric Lévy distribution is a stretched exponential function, we obtain:

$$\Phi(t)=\exp[-A(\lambda t^{1/\alpha})^\alpha]=\exp(-A\lambda^\alpha t). \tag{41}$$

This means that the exponential relaxation under Lévy dynamics remains a simple exponential relaxation (only the corresponding relaxation time changes). For example, the first moment of the distribution (the particle's position) still relaxes exponentially to its equilibrium value of zero. On the other hand, the dependence of the relaxation time on the outer parameters (say, temperature) entering through the values of $\gamma$ and $D$ can change considerably. Thus the superdiffusive Lévy-flight dynamics in the force-free case can coexist with standard thermodynamics and with very simple relaxation patterns, as far as the case of a harmonic force is concerned.

Let us consider the relaxation in a harmonic potential under ''paradoxical'' diffusion. Here again we can use the moment expansion [Eq. (39)], and set down an expression for the characteristic function of the overall distribution:

$$P(k,t)=\int\exp[-ikx'e^{-\gamma n}-Dk^2(1-e^{-2\gamma n})/2\gamma]\times S_\alpha(n,t)dn. \tag{42}$$

Note that the moments of the corresponding distribution are combinations of the functions:

$$\Phi(t)=\int_0^\infty\exp(-\lambda T)S_\alpha(T,t)dT. \tag{43}$$

Note that $S_\alpha(n,t)$ is a PDF of a process subordinated to a Lévy distribution under a TET:

$$S_\alpha(T,t)=\int d\tau\tau^{-1/\alpha}L_\alpha(T/\tau^{1/\alpha},\alpha,-\alpha)Q_\alpha(\tau,t)d\tau. \tag{44}$$

Thus a Laplace transform of $S$, according to its outer time variable, is a stretched-exponential, so that

$$\Phi(t)=\int_0^\infty p(\tau,t)\exp(-A\lambda^\alpha\tau)d\tau. \tag{45}$$

Let us take a Laplace transform of this expression. Using Eq. (26), we obtain

$$\Phi(u)=\int_0^\infty u^{\alpha-1}\exp(-\tau u^\alpha)\exp(-A\lambda^\alpha\tau)d\tau=\frac{u^{\alpha-1}}{u^\alpha+A\lambda^\alpha}. \tag{46}$$

For small $u$ (long times) this corresponds to a power-law decay of $\Phi(t)$ of a form $\Phi(t)\propto t^{-\alpha}$ for $t\gg\lambda^{-1}$. Thus the relaxation in the case of paradoxical diffusion resembles those in normal CTRW, and is dominated by large lacunae. In the case when the processes are subordinated the other way around, i.e., according to $S_\alpha^*(T,t)$, the decay at longer times follows the universal $t^{-1}$-law: for example, for $\alpha=1/2$, we obtain

$$\Phi(t)=\frac{2t}{\pi}\int_0^\infty\exp(-\lambda T)\frac{1}{t^2+T^2}dT=\frac{2\lambda}{\pi}[\sin(\lambda t)\mathrm{ci}(\lambda t)-\cos(\lambda t)\mathrm{si}(\lambda t)], \tag{47}$$

see Eq. (2.3.7.11) of Ref. [25] [here the integral sine and cosine functions, $\mathrm{si}(x)=-\int_x^\infty(\sin x/x)dx$ and $\mathrm{ci}(x)=-\int_x^\infty(\cos x/x)dx$, are used]. For $\lambda t\gg 1$, we obtain

$$\Phi(t)\simeq\frac{2}{\pi}(\lambda t)^{-1}, \tag{48}$$

the asymptotic behavior which is universal for all Lévy-driven CTRW's of the same index.

## IX. CONCLUSIONS

A broad range of physical processes can be described as processes subordinated to a random walk under some operational time. In particular, such a subordination leads to anomalous transport properties, a well-known example being the Scher-Montroll continuous-time random walks, a process in which the operational time (given by the number of steps) is sublinear in the physical time *t*. Here we have considered the processes subordinated to a diffusive process under an operational time governed by a Lévy distribution with index $0<\alpha<1$; that is, the operational time is superlinear in physical one. We have shown that in the absence of outer forces this subordination leads exactly to Lévy flights. The response of such a system to a weak outer force is strongly nonlinear. Interestingly enough, the relaxation patterns in such systems are simpler than expected. Thus we show that the behavior in the presence of a weak harmonic force corresponds to a simple exponential relaxation to a normal Boltzmann distribution. The combination of superlinear- and sublinear operational times (i.e., Lévy flights under a sublinear operational time or a Scher-Montroll CTRW under Lévy time) corresponds to ''paradoxical'' diffusion, a random process which in a force-free case leads to probability distributions of the particle's displacements, which show the power-law tails and lack a second moment. The width of the distribution, on the other hand, grows proportionally to the square root of the time, showing a typically diffusive behavior. Some physical implications of these findings have been discussed.

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